On the Existence of Radiation Gauges in Petrov type II spacetimes

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Abstract. The radiation gauges used by Chrzanowski (his IRG/ORG) for metric reconstruction in the Kerr spacetime seem to be over-specified. Their specification consists of five conditions: four (which we treat here as) "gauge" conditions plus an additional condition on the trace of the metric perturbation. In this work, we utilize a newly developed form of the perturbed Einstein equations to establish a condition — on a particular tetrad component of the stress-energy tensor — under which one can impose the full IRG/ORG. In a Petrov type II background, imposing the IRG/ORG additionally requires (consistently) setting a particular component of the metric perturbation to zero "by hand". By contrast, in a generic type D background, gauge freedom can generally be used to achieve this. As a specific example, we work through the process of imposing the IRG in a Schwarzschild background, using a more traditional approach. Implications for metric reconstruction using the Teukolsky curvature perturbations in type D spacetimes are briefly discussed.

1. Introduction

The Regge-Wheeler[20] (RW) approach to perturbations of the Schwarzschild spacetime is usually understood to lead, by direct integration, to perturbations for all parts of the metric in terms of gauge invariant quantities[13]. In fact, it has been evident for a long time[19] that the RW variable actually represents part of the perturbation of the Weyl curvature (namely, $\mathrm{Im}(\Psi_2)$[15]; see [9] for a clear demonstration of this and [25] for further discussion). These contrasting perspectives are reconciled by the exceptional fact that the geometrical symmetries of the Schwarzschild spacetime permit an analysis virtually transparent to both angular and time derivatives.

For the perturbations of the Kerr spacetime, the situation is completely different. Instead of the RW equation, we have the Teukolsky equation for the gauge (and tetrad) invariant parts (Ψ_0 and Ψ_4) of the perturbed Weyl curvature. To date, metric reconstruction[16, 12] can then be obtained by a (Hertz) potential method championed by Chrzanowski[1], but only fully in vacuum, and even so, only in a special class of over-specified gauges referred to as "radiation gauges". We expect that Hertz potential methods are set to play an increasingly key rôle as we pursue deeper studies of perturbations of Petrov type D spacetimes. However, neither Chrzanowski's analysis, nor the more general analyses of Cohen and Kegeles[2, 10] and Stewart[22] for perturbations of Petrov type II spacetimes, spells out the precise circumstances in which radiation gauges are able to exist.† The purpose of this paper is to address and dispel this concern, by specifying exactly when a radiation gauge may be imposed.

The Petrov classification refers to the properties of null eigenvectors of the Weyl tensor, referred to as principle null directions (PNDs). In a type II spacetime, one of these PNDs is repeated. It may be either the ingoing null vector l^a or the outgoing null vector n^a. In a type D spacetime, two of the PNDs are repeated, namely, both l^a and n^a. Radiation gauges have been defined[1] with respect to either one of these PNDs. However, the gauge conditions specified for a radiation gauge are either $l^a h_{ab} = 0$ and $g^{ab} h_{ab} = 0$, referred to as "Ingoing" (IRG), or $n^a h_{ab} = 0$ and $g^{ab} h_{ab} = 0$, referred to as "Outgoing" (ORG). In each case, since these represent five distinct conditions, it is clear that radiation gauges cannot be defined in general, but it turns out they can prescribed in special circumstances, which we have investigated here.

In general terms, in type D spacetimes, two radiation gauges are indeed possible. In type II spacetimes, depending on which principle null direction is repeated, only one or the other of these gauges would be possible. In all cases, we find that radiation gauges can normally actually exist only for perturbations with $\mathcal{T}_{ll} = 0$, for the equation for the trace of the metric perturbation requires no source in order for it to have a zero solution. In all type D spacetimes, a non-zero solution to the trace equation without sources can generally be gauged away by the use of residual gauge freedom. As far as we can tell without further analysis, the zero solution to the trace equation must be *chosen* in the type II case in order to ensure the radiation gauge conditions are fully satisfied.

† However, the constructive procedure of Stewart does go a long way in this direction.

The layout of the paper is as follows. We first introduce a new form of the perturbed Einstein equations in the Newman-Penrose formalism. Then, we describe the radiation gauges in more detail, followed by explanations of how they are set up in type II and type D spacetimes, respectively. This requires us to examine the perturbed Einstein tensor to understand fully the implications of attempting to impose a radiation gauge. We illustrate the residual gauge freedom and the condition for it to remove the trace of the perturbed metric. Next, we demonstrate the implication of our analysis in Schwarzschild spacetime, and discuss issues particular to spin-zero perturbations. Finally we include a discussion relating the existence of Hertz potentials to the existence of radiation gauges.

2. A new form of the perturbed Einstein equations

We choose to work with a formulation of the perturbed Einstein equations that makes explicit use of the modified Newman-Penrose[14] (NP) formalism of Geroch, Held and Penrose[4] (GHP). For a detailed explanation of the GHP formalism see also [17]. The starting point is to take a complex null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ normalized so that‡

$$l^a n_a = -m^a \bar{m}_a = 1. \tag{1}$$

Then, the spacetime metric has the following expression:

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}, \tag{2}$$

in which round brackets, (), around indicies denotes symmetrization. We can express the metric perturbation, h_{ab}, in terms of the tetrad vectors according to

$$\begin{aligned} h_{ab} = {} & h_{nn} l_a l_b - 2h_{n\bar{m}} l_{(a}m_{b)} - 2h_{nm} l_{(a}\bar{m}_{b)} + 2h_{ln} l_{(a}n_{b)} \\ & + h_{ll} n_a n_b - 2h_{l\bar{m}} n_{(a}m_{b)} - 2h_{lm} n_{(a}\bar{m}_{b)} \\ & + h_{mm}\bar{m}_a\bar{m}_b + 2h_{m\bar{m}} m_{(a}\bar{m}_{b)} + h_{\bar{m}\bar{m}} m_a m_b, \end{aligned} \tag{3}$$

where $h_{ll} = h_{ab}l^a l^b$, $h_{lm} = h_{ab}l^a m^b$ and so on, are the tetrad components of the metric perturbation. The perturbed Einstein equations are then computed via

$$\mathcal{E}_{ab} = -\tfrac{1}{2}\Theta^c\Theta_c h_{ab} - \tfrac{1}{2}\Theta_a\Theta_b h^c{}_c + \Theta^c\Theta_{(a}h_{b)c} + \tfrac{1}{2}g_{ab}(\Theta^c\Theta_c h^d{}_d - \Theta^c\Theta^d h_{cd}), \tag{4}$$

where

$$\Theta_a = l_a Þ' + n_a Þ - m_a ð' - \bar{m}_a ð, \tag{5}$$

is the just the covariant derivative expressed in GHP language. We can use the expression in (5) to define the GHP derivatives 'thorn' ($Þ = l^a\Theta_a$), 'edth' ($ð = m^a\Theta_a$) and their 'primes' ($Þ' = n^a\Theta_a$ and $ð' = \bar{m}^a\Theta_a$). We will refer to (4) as the GHP form of the Einstein equations, or simply the EEs. The EEs for a generic Petrov type II spacetime are given in Appendix A.

‡ The conventions displayed in (1) and (2) with signature [+,-,-,-] are characteristic of the NP formalism, but differ from those used in other parts of general relativity, including section 6 and Appendix C.

3. The Radiation Gauges

The ingoing radiation gauge (IRG) is a crucial ingredient for the reconstruction of metric perturbations of Petrov type D spacetimes from curvature perturbations. They first appear in the work of Cohen and Kegeles [2] (for perturbations of Petrov type II spacetimes) and Chrzanowski [1] (who considered perturbations of Petrov type D spacetimes), but the work that comes closest to our contribution in describing their origin is that of Stewart [22], again for the more general case of type II spacetimes.

In type II spacetimes, the IRG is defined by the conditions

$$l^a h_{ab} = 0, \tag{6a}$$

$$g^{ab} h_{ab} = 0, \tag{6b}$$

where l^a is aligned with the repeated PND of the background Weyl tensor. If n^a rather than l^a is a repeated PND, we can instead define the outgoing radiation gauge (ORG) by

$$n^a h_{ab} = 0, \tag{7a}$$

$$g^{ab} h_{ab} = 0. \tag{7b}$$

In type II spacetimes, only one or the other of these options exists (IRG or ORG), whereas in Petrov type D spacetimes, there is the possibility of defining both gauges. In the first part of this work we will restrict our attention to an IRG in the more general case of a Petrov type II background.

Equations (6) translate into algebraic conditions on the components of the metric perturbation. We will refer to the four conditions in (6a) as "gauge" conditions.§ In terms of the tetrad components of the metric perturbation, the gauge conditions read:

$$\begin{aligned} h_{ll} &= 0, \\ h_{ln} &= 0, \\ h_{lm} &= 0, \\ h_{l\bar{m}} &= 0. \end{aligned} \tag{8}$$

The condition in (6b) will be referred to as the trace condition and can be expressed in terms of the components of the metric perturbation as $h_{ln} - h_{m\bar{m}} = 0$, which, when (8) is imposed, simply reads

$$h_{m\bar{m}} = 0. \tag{9}$$

It is useful to note the similarity between the full IRG (6) and the more commonly known transverse traceless (TT) gauge defined by

$$\begin{aligned} \nabla^a h_{ab} &= 0, \\ g^{ab} h_{ab} &= 0, \end{aligned} \tag{10}$$

§ Recently, when applied specifically to the Schwarzschild spacetime, these conditions were given a geometrical interpretation, and referred to as *light-cone gauge conditions*[18]. It may well be that this description is suitable more generally, although presumably without the geometrical interpretation.

which, at a first glance, also appears to be over-specified. In fact, the TT gauge exists for any vacuum perturbation of an arbitrary, globally hyperbolic, vacuum solution[23], because imposing the differential part of gauge does not exhaust all of the available gauge freedom. Interestingly enough, Stewart's analysis in terms of Hertz potentials[22] begins by considering a metric perturbation in the TT gauge. However, in order to construct the curved space analogue of a Hertz potential, he is forced to perform a transformation that destroys (10) and instead yields a metric perturbation in the IRG.‖ Furthermore it appears that the restriction to type II spacetimes is essential for Stewart's analysis. From these observations, we expect radiation gauges to exist under conditions less general than those required for the existence of the TT gauge. At the same time, we should not be surprised that the IRG inherits the feature of *residual gauge freedom*.

Consider a gauge transformation on the metric perturbation generated by a gauge vector, ξ_a. To create a transformed metric in the IRG, the "gauge" conditions (8) require

$$l^a(h_{ab} - \xi_{(a;b)}) = 0, \tag{11}$$

where the semicolon denotes the covariant derivative. In terms of components this reads

$$\begin{aligned}
&2\text{Þ}\xi_l &&= h_{ll},\\
&\text{Þ}'\xi_l + \text{Þ}\xi_n + (\tau + \bar{\tau}')\xi_{\bar{m}} + (\bar{\tau} + \tau')\xi_m &&= h_{ln},\\
&(\text{Þ} + \bar{\rho})\xi_m + (\eth + \bar{\tau}')\xi_l &&= h_{lm},\\
&(\text{Þ} + \rho)\xi_{\bar{m}} + (\eth' + \tau')\xi_l &&= h_{l\bar{m}}.
\end{aligned} \tag{12}$$

Similarly, for the "trace" condition to be satisfied by the gauge transformed metric, we require

$$\eth'\xi_m + \eth\xi_{\bar{m}} + (\rho' + \bar{\rho}')\xi_l + (\rho + \bar{\rho})\xi_n = h_{m\bar{m}}. \tag{13}$$

Any extra gauge transformation that satisfies $l^a\xi_{(a;b)} = 0$, that is, solves the homogeneous form of (12), preserves the four IRG "gauge" conditions (8). This is what is meant by residual gauge freedom. When we consider the case of an arbitrary type D background spacetime, we will explicitly use this residual gauge freedom to determine when we can impose the IRG "gauge" and "trace" conditions simultaneously. First we turn our attention to the general case of type II spacetimes.

4. Imposing the IRG in type II

In a spacetime more general than type II, there is no possibility of having a repeated PND. When a repeated does PND exists, we can appeal to the Golberg-Sachs theorem [6] and set $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ in the EEs. Then we are in a position to address the question of when the full IRG can be imposed. First we apply the four gauge conditions (8) to the EEs. While most of the EEs depend on several components of the metric perturbation, the equation for $\mathcal{E}_{ll}$ depends only on $h_{m\bar{m}}$ and simply becomes

$$\{\text{Þ}(\text{Þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}} \equiv \{(\text{Þ} - 2\rho)(\text{Þ} + \rho - \bar{\rho})\}h_{m\bar{m}} = 8\pi\mathcal{T}_{ll}, \tag{14}$$

‖ In flat space, owing to the fact that partial derivatives commute, this transformation would actually leave one in the TT gauge. See [22] or Appendix C of [25] for a more detailed explanation.

in which the first form indicates that the equation is real, while the second form and its complex conjugate are used in the integrations of Appendix B. If we had not made use of the Goldberg-Sachs theorem, there would be terms such as $\sigma\rho h_{\bar{m}\bar{m}}$ appearing in (14) and our argument would not hold. We see that if $\mathcal{T}_{ll} \neq 0$, (9) cannot hold, whereas if $\mathcal{T}_{ll} = 0$, it is at least possible to impose (9) as a solution of (14).¶ The condition $\mathcal{T}_{ll} = 0$ is a necessary condition for an IRG to exist. Whether the implied condition, $\mathcal{E}_{ll} = 0$, is also sufficient, remains to be investigated. We can go no further for type II spacetimes.

5. Imposing the IRG in type D

Our situation improves greatly when we restrict our attention to type D backgrounds. These are of considerable theoretical and observational interest since they include both the Schwarzschild and Kerr (rotating) black hole spacetimes. Kinnersley first obtained all type D metrics by integrating the Newman-Penrose equations[11]. Following that, Held introduced a method for performing a coordinate-free integration of the GHP equations. We will make heavy use of Held's method here to solve both $\mathcal{E}_{ll} = 0$ and $l^a\xi_{(a;b)} = 0$, the homogeneous form of (12). First, we will review Held's method. Rather than give a detailed explanation, we present the basics and refer the interested reader to the literature for an in-depth account[7, 21].

The first step is to introduce new derivative operators $\tilde{\text{Þ}}'$, $\tilde{\eth}$ and $\tilde{\eth}' = \bar{\tilde{\eth}}$ such that they commute with Þ when acting on quantities that Þ annihilates,+ that is

$$
\begin{aligned}
[\text{Þ}, \tilde{\text{Þ}}']x^\circ &= 0, \\
[\text{Þ}, \tilde{\eth}\,]x^\circ &= 0, \\
[\text{Þ}, \tilde{\eth}']x^\circ &= 0,
\end{aligned}
\tag{15}
$$

where $[a, b]$ dentoes the commutator between a and b. The explicit form of the operators is given in Appendix B. The next step, the heart of Held's method, is to exploit the GHP equation $\text{Þ}\rho = \rho^2$, and its complex conjugate $\text{Þ}\bar{\rho} = \bar{\rho}^2$, to express everything as a polynomial in terms of ρ and $\bar{\rho}$, with coefficeints that are annihilated by Þ. Held's method is then brought to completion by choosing four independent quantities to use as coordinates[8, 3]. We will not need to take this final step because our result can be established once everything is expressed as a polynomial in ρ and $\bar{\rho}$.

Full integration of the equations is carried out in the Appendix B. The result of integrating $\mathcal{E}_{ll} = 0$ is

$$
h_{m\bar{m}} = a^\circ \frac{\rho}{\bar{\rho}} + \bar{a}^\circ \frac{\bar{\rho}}{\rho} + b^\circ(\rho + \bar{\rho}),
\tag{16}
$$

where $b^\circ = \bar{b}^\circ$. Integration of the homogenous form of (12) leads to the following general

¶ The general solution is derived in Appendix B and given in (16).
+ Such quantities are denoted with the degree mark, $^\circ$, as in $\text{Þ}x^\circ = 0$.

solution for the components of the gauge vector:

$$\begin{aligned}
\xi_l \; &= {\xi_l}^\circ, \\
\xi_n \; &= {\xi_n}^\circ + \tfrac{1}{2}\Psi^\circ {\xi_l}^\circ \rho + \tfrac{1}{2}\bar{\Psi}^\circ {\xi_l}^\circ \bar{\rho} + \tau^\circ \bar{\tau}^\circ {\xi_l}^\circ \rho\bar{\rho} + \tfrac{1}{2}\pi^\circ \bar{\pi}^\circ {\xi_l}^\circ \left(\frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2}\right) \\
&\quad + \left[\frac{\pi^\circ}{\rho}(\tilde{\eth} + \bar{\alpha}^\circ) + \frac{\bar{\pi}^\circ}{\bar{\rho}}(\tilde{\eth}' + \alpha^\circ)\right]{\xi_l}^\circ + \tfrac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\text{Þ}' {\xi_l}^\circ \\
&\quad - [\bar{\tau}^\circ \rho(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ \bar{\rho}(\tilde{\eth}' + \alpha^\circ)]{\xi_l}^\circ + \bar{\tau}^\circ {\xi_m}^\circ \frac{\rho}{\bar{\rho}} + \tau^\circ {\xi_{\bar{m}}}^\circ \frac{\bar{\rho}}{\rho} \\
&\quad - \pi^\circ {\xi_m}^\circ \frac{1}{\bar{\rho}^2} - \bar{\pi}^\circ {\xi_{\bar{m}}}^\circ \frac{1}{\rho^2} - \alpha^\circ {\xi_m}^\circ \frac{1}{\bar{\rho}} - \bar{\alpha}^\circ {\xi_{\bar{m}}}^\circ \frac{1}{\rho}, \\
\xi_m &= {\xi_m}^\circ \frac{1}{\bar{\rho}} - \bar{\pi}^\circ {\xi_l}^\circ \frac{1}{\rho} + \tau^\circ {\xi_l}^\circ \bar{\rho} - (\tilde{\eth} + \bar{\alpha}^\circ){\xi_l}^\circ, \\
\xi_{\bar{m}} &= {\xi_{\bar{m}}}^\circ \frac{1}{\rho} - \pi^\circ {\xi_l}^\circ \frac{1}{\bar{\rho}} + \bar{\tau}^\circ {\xi_l}^\circ \rho - (\tilde{\eth}' + \alpha^\circ){\xi_l}^\circ,
\end{aligned} \tag{17}$$

where the quantities Ψ°, τ°, π° and α° determine properties of the background spacetime.* While the general form of the gauge vector (17) is rather complicated, for (most of) our purposes it is sufficient to consider the equations with $\xi_l = {\xi_l}^\circ = 0$. In that case (13) becomes, after some manipulation (using (B.1) and (B.6-B.8) in Appendix B),

$$h_{m\bar{m}} = \frac{\rho}{\bar{\rho}}\tilde{\eth}' {\xi_m}^\circ + \frac{\bar{\rho}}{\rho}\tilde{\eth}{\xi_{\bar{m}}}^\circ + (\rho + \bar{\rho}){\xi_n}^\circ. \tag{18}$$

Comparison with (16) suggests that, in type D, for $h_{m\bar{m}}$ a non-zero solution of $\mathcal{E}_{ll} = 0$ given by (16), we can generally exploit the residual gauge freedom to obtain $h_{m\bar{m}} = 0$ (without affecting $\mathcal{E}_{ll} = 0$). Generally, a gauge vector with

$$\begin{aligned}
{\xi_l}^\circ \quad &= 0, \\
{\xi_n}^\circ \quad &= b^\circ, \\
\tilde{\eth}{\xi_m}^\circ \; &= a^\circ, \\
\tilde{\eth}' {\xi_{\bar{m}}}^\circ &= \bar{a}^\circ,
\end{aligned} \tag{19}$$

will achieve this. The requirement $\mathcal{T}_{ll} = 0$ is both necessary and sufficient for imposing an IRG in type D spacetimes, in contrast to the situation in type II, where we must impose $h_{m\bar{m}} = 0$ by hand, as a solution of $\mathcal{E}_{ll} = 0$, to ensure the existence of an IRG.

6. Imposing the IRG in Schwarzschild spacetime♯

We now demonstrate these results for Schwarzschild spacetime using conventional, spherically symmetric coordinates, in which the background metric takes the form:

$$ds^2 = -f(r)\, dt^2 + dr^2/f(r) + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{20}$$

* For example, $\pi^\circ \neq 0$ leads to the accelerating C-metrics. The condition $\pi^\circ = 0$ implies $\alpha^\circ = 0$ and so α° is also related to parameters in the C-metric.

♯ In this entire section and in Appendix C, equations (1) and (2) are replaced by $-l^a n_a = m^a \bar{m}_a = 1$ and $g_{ab} = -2l_{(a}n_{b)} + 2m_{(a}\bar{m}_{b)}$ respectively, and the metric signature is [-,+,+,+].

in which we have introduced $f(r)$=$(1-2M/r)$. Metric perturbations, $h_{ab}(t,r,\theta,\phi)$, about the Schwarzschild geometry can be expressed in terms of a RW decomposition[20]. We decompose the angular dependence of these perturbations into spherical harmonics and the time dependence into constant frequency Fourier modes:

$$h_{ab}(t,r,\theta,\phi) = \sum_{lm\omega} e^{-i\omega t}\, h_{ab}^{lm\omega}(r) Y_{\ell m}(\theta,\phi). \tag{21}$$

With respect to rotation of the background coordinate system, h_{tt}, h_{rr} and h_{tr} transform as scalars, $\{h_{t\theta},\, h_{t\phi}\}$ and $\{h_{r\theta},\, h_{r\phi}\}$ transform as a pair of vectors on the 2-sphere and $\{h_{\theta\theta},\, h_{\theta\phi},\, h_{\phi\phi}\}$ transforms as a symmetric covariant tensor on the 2-sphere.

It is well known the the components of the metric perturbation decouple into two classes, labelled as even and odd parity, according to their behavior under a parity transformation P :$(\theta \to \pi - \theta,\, \phi \to \pi + \phi)$. Out of the ten independent components of the metric perturbation for each mode (of specific $lm\omega$), the even parity perturbations, for which $P = (-1)^l$, have seven independent components and are given by $h_{ab}^{\rm even}$,

$$e^{-i\omega t}\begin{bmatrix} f(r)H_0(r) & H_1(r) & h_0(r)\frac{\partial}{\partial\theta} & h_0(r)\frac{\partial}{\partial\phi} \\ \text{sym} & H_2(r)/f(r) & h_1(r)\frac{\partial}{\partial\theta} & h_1(r)\frac{\partial}{\partial\phi} \\ \text{sym} & \text{sym} & r^2\left[K(r)+G(r)\frac{\partial^2}{\partial\theta^2}\right] & r^2G(r)\left[\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right] \\ \text{sym} & \text{sym} & \text{sym} & \begin{array}{c} r^2G(r)\left[\frac{\partial^2}{\partial\phi^2} + \frac{\sin 2\theta}{2}\frac{\partial}{\partial\theta}\right] \\ +r^2K(r)\sin^2\theta \end{array} \end{bmatrix} Y_{\ell m}(\Omega).$$

The odd parity perturbations, with $P = (-1)^{l+1}$, have three independent components, and are given by $h_{ab}^{\rm odd}$,

$$e^{-i\omega t}\begin{bmatrix} 0 & 0 & -h_0(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_0(r)\sin\theta\frac{\partial}{\partial\theta} \\ 0 & 0 & -h_1(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & h_1(r)\sin\theta\frac{\partial}{\partial\theta} \\ \text{sym} & \text{sym} & h_2(r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\right] & \begin{array}{c} \frac{1}{2}h_2(r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} + \right. \\ \left. \cos\theta\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial^2}{\partial\theta^2}\right] \end{array} \\ \text{sym} & \text{sym} & \text{sym} & -h_2(r)\left[\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi}\right] \end{bmatrix} Y_{\ell m}(\Omega).$$

In the background spacetime, the Einstein tensor is identically zero. The perturbed Einstein tensor (which includes contributions from the metric perturbation h_{ab} up to first order) transforms in the same way as the metric perturbations with respect to rotations on the 2-sphere. Hence, it has the same angular decomposition as the metric perturbation. For the even parity, we write the perturbed Einstein tensor, $G_{ab}^{\rm even}$, as

$$e^{-i\omega t}\begin{bmatrix} f(r)E_1(r) & E_2(r) & E_4(r)\frac{\partial}{\partial\theta} & E_4(r)\frac{\partial}{\partial\phi} \\ \text{sym} & E_3/f(r) & E_5(r)\frac{\partial}{\partial\theta} & E_5(r)\frac{\partial}{\partial\phi} \\ \text{sym} & \text{sym} & r^2\left[E_6(r)+E_7(r)\frac{\partial^2}{\partial\theta^2}\right] & r^2E_7(r)\left[\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right] \\ \text{sym} & \text{sym} & \text{sym} & \begin{array}{c} r^2E_7(r)\left[\frac{\partial^2}{\partial\phi^2} + \frac{\sin 2\theta}{2}\frac{\partial}{\partial\theta}\right] \\ +r^2E_6(r)\sin^2\theta \end{array} \end{bmatrix} Y_{\ell m}(\Omega),$$

and for odd parity we write perturbed Einstein tensor, $G^{\rm odd}_{ab}$, as

$$e^{-i\omega t}\begin{bmatrix} 0 & 0 & -F_1(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & F_1(r)\sin\theta\frac{\partial}{\partial\theta} \\ 0 & 0 & -F_2(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & F_2(r)\sin\theta\frac{\partial}{\partial\theta} \\ \text{sym} & \text{sym} & F_3(r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\right] & \begin{array}{c}\frac{1}{2}F_3(r)\left[\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}+\right.\\ \left.\cos\theta\frac{\partial}{\partial\theta}-\sin\theta\frac{\partial^2}{\partial\theta^2}\right]\end{array} \\ \text{sym} & \text{sym} & \text{sym} & -F_3(r)\left[\sin\theta\frac{\partial^2}{\partial\theta\partial\phi}-\cos\theta\frac{\partial}{\partial\phi}\right] \end{bmatrix} Y_{\ell m}(\Omega).$$

The spherical symmetry and the static nature of the background geometry ensures that the perturbed Einstein equations decouple into individual modes of $\{\ell,m,\omega,P\}$. That is, each component of the perturbed G_{ab} belonging to a specific $\{\ell,m,\omega,P\}$ mode depends only on the metric perturbations of the same mode, $h^{\ell m\omega P}_{ab}$. Hence, it is generally sufficient to consider a single mode of the metric perturbation for our analysis.

We now impose the gauge conditions $(6a)$ on a specific mode of the perturbed metric. We have, $l^a h_{ab}=0$, where $l^a=(1/f(r),1,0,0)$ from equations (39) below, and l^a is a repeated PND of the Schwarzschild background. For the odd parity perturbations, we can write h_0 in terms of h_1.

$$h_0(r) = -f(r)h_1(r). \tag{22}$$

For the even parity perturbations,

$$H_0(r) = -H_1(r) = H_2(r), \quad \text{and} \quad h_0(r) = -f(r)h_1(r). \tag{23}$$

The trace of the metric perturbations is a scalar with respect to rotation on a sphere of constant r and constant t. Hence, it can be written as

$$h_{ab}g^{ab} = \sum_{\ell m\omega P} e^{-i\omega t}\, T^{\ell m\omega P}(r)\, Y_{\ell m}(\theta,\phi), \tag{24}$$

Expanding the LHS gives,

$$h_{ab}g^{ab} = -\frac{1}{f(r)}h_{tt} + f(r)h_{rr} + \frac{1}{r^2}h_{\theta\theta} + \frac{1}{r^2\sin^2\theta}h_{\phi\phi}. \tag{25}$$

Note that the trace vanishes for the odd parity perturbations, while for the even parity the trace is equal to (we suppress labels $\{\ell m\omega P\}$ when ambiguity is unlikely)

$$T(r) = \left[2K(r) - \ell(\ell+1)G(r)\right]. \tag{26}$$

We have used the fact that the spherical harmonics are eigenfunctions of the angular momentum operator:

$$\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right]Y_{\ell m}(\theta,\phi) = -\ell(\ell+1)Y_{\ell m}(\theta,\phi). \tag{27}$$

The perturbed Einstein tensor obtained in this gauge is given in Appendix C.

6.1. Residual gauge freedom in Schwarzschild

To determine the residual gauge freedom in Schwarzschild, we require first, as in (11),

$$l^a h_{ab} = l^a \xi_{(a;b)}. \tag{28}$$

Writing $l^a h_{ab}$ as B_b and using the metric (20) to compute the covariant derivatives, gives

$$\begin{aligned} B_t =& [2\xi_{t,t} - f'(r)\xi_t]/f(r) + \xi_{r,t} + \xi_{t,r} - f'(r)\xi_r, \\ B_r =& [\xi_{t,r} + \xi_{r,t} + f'(r)\{\xi_r - \xi_t/f(r)\}]/f(r) + 2\xi_{r,r}, \\ B_\theta =& [\xi_{t,\theta} + \xi_{\theta,t}]/f(r) + \xi_{r,\theta} + \xi_{\theta,r} - 2\xi_\theta/r, \\ B_\phi =& [\xi_{t,\phi} + \xi_{\phi,t}]/f(r) + \xi_{r,\phi} + \xi_{\phi,r} - 2\xi_\phi/r. \end{aligned} \tag{29}$$

Gauge vectors ξ_a, which correspond to residual gauge freedom, solve the above equations with $B_a = 0$. Moreover, since we are going to deal with the metric perturbations of one single mode (specific ℓ, m, ω, P) at a time, we want h_{ab} and ξ_a to correspond to the same mode. This restricts the functional form of our gauge vector $\xi_a(x^b)$.

For even parity perturbations, we consider an even gauge vector of the form,

$$\begin{aligned} \xi_t =& - e^{-i\omega t}\, P(r) Y_{\ell m}(\theta,\phi), \\ \xi_r =& e^{-i\omega t}\, R(r) Y_{\ell m}(\theta,\phi), \\ \xi_\theta =& e^{-i\omega t}\, rS(r) \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\phi), \\ \xi_\phi =& e^{-i\omega t}\, rS(r) \frac{\partial}{\partial\phi} Y_{\ell m}(\theta,\phi). \end{aligned} \tag{30}$$

For odd parity perturbations, we consider an odd gauge vector of the form,

$$\begin{aligned} \xi_t =& 0, \\ \xi_r =& 0, \\ \xi_\theta =& - e^{-i\omega t}\, Q(r) \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} Y_{\ell m}(\theta,\phi), \\ \xi_\phi =& e^{-i\omega t}\, Q(r) \sin\theta \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\phi). \end{aligned} \tag{31}$$

Inserting these forms of gauge vector into (29) and taking $B_a = 0$, we arrive at equations for residual gauge freedom. For the even parity gauge vector, we have

$$\begin{aligned} 0 =& e^{-i\omega t} \left[\{2i\omega P(r) + f'(r)P(r)\}/f(r) - i\omega R(r) - P'(r) - f'(r)R(r)\right] Y_{\ell m}(\theta,\phi), \\ 0 =& e^{-i\omega t} \left[\{f'(r)[R(r) + P(r)/f(r)] - i\omega R(r) - P'(r)\}/f(r) + 2R'(r)\right] Y_{\ell m}(\theta,\phi), \\ 0 =& e^{-i\omega t} \left[-\{i\omega r S(r) + P(r)\}/f(r) + R(r) - S(r) + rS'(r)\right] \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\phi), \\ 0 =& e^{-i\omega t} \left[-\{i\omega r S(r) + P(r)\}/f(r) + R(r) - S(r) + rS'(r)\right] \frac{\partial}{\partial\phi} Y_{\ell m}(\theta,\phi). \end{aligned} \tag{32}$$

For the odd parity gauge vector, we have

$$\begin{aligned} 0 =& e^{-i\omega t} \left[i\omega Q(r)/f(r) - Q'(r) + 2Q(r)/r\right] \frac{1}{\sin\theta} \frac{\partial}{\partial\phi} Y_{\ell m}(\theta,\phi), \\ 0 =& -e^{-i\omega t} \left[i\omega Q(r)/f(r) - Q'(r) + 2Q(r)/r\right] \sin\theta \frac{\partial}{\partial\theta} Y_{\ell m}(\theta,\phi). \end{aligned} \tag{33}$$

These equations can be solved completely for the functions $P(r)$, $R(r)$, $S(r)$ in terms of three arbitrary constants (we have introduced $r_* = r + 2M\ln(r/2M - 1)$):

$$\begin{aligned} P(r) &= e^{i\omega r_*}\left[C_1 - C_2\left(i\omega r + f(r)\right)\right], \\ R(r) &= e^{i\omega r_*}\left[C_1 - i\omega r C_2\right]/f(r), \\ S(r) &= e^{i\omega r_*}\left[C_2 + C_3 r\right]. \end{aligned} \tag{34}$$

The function $Q(r)$ can be solved in terms of one arbitrary constant:

$$Q(r) = De^{i\omega r_*}r^2. \tag{35}$$

6.2. Condition for the Trace to vanish

The residual gauge freedom can be used to change the trace of the metric perturbation by a quantity $\mathrm{Tr}(\xi_{(a;b)}) = g^{ab}\xi_{(a;b)}$. For an odd parity perturbation, this quantity is easily seen to be zero. For an even parity perturbation, this quantity is evaluated to be

$$\mathrm{Tr}(\xi_{(a;b)}) = -e^{-i\omega(t-r_*)}\left(2i\omega C_2 + \ell(\ell+1)C_3 - \frac{2C_1 - \ell(\ell+1)C_2}{r}\right)Y_{\ell m}(\theta,\phi). \tag{36}$$

A particular linear combination, $E_1 + 2E_2 + E_3$, of the Einstein tensor components given in Appendix C is exactly $f(r)\mathcal{E}_{ll}$. For perturbations satisfying $\mathcal{T}_{ll} = 0$, this condition becomes a second order differential equation acting only on the variable $T(r)$:

$$2\mathcal{E}_{ll} = T''(r) + \frac{2(-i\omega r^2 + r - 2M)}{r(r-2M)}T'(r) + \frac{-2i\omega(r-3M) - \omega^2 r^2}{(r-2M)^2}T(r) = 0. \tag{37}$$

The solution to this equation is obtained in terms of two arbitrary constants A, B:

$$T(r) = e^{i\omega r_*}(A + B/r). \tag{38}$$

From (36), we already know the degrees of freedom that exist in the trace of the metric perturbation due to residual gauge freedom. One sees that, generally, the arbitrary constants C_1, C_2, C_3 can be chosen to exactly cancel A and B. Thus, this analysis confirms that the residual gauge freedom can generally be exploited to set the trace of the metric perturbations to be zero for perturbations with $\mathcal{T}_{ll} = 0$, and verifies that $\mathcal{E}_{ll} = 0$ is a sufficient condition for constructing an IRG. Once this has been done, there still exists one constant residual degree of gauge freedom per mode of metric perturbation (both even and odd). It is not clear how to fix these degrees of freedom in order to get some more useful analytical property of the metric perturbations.

6.3. Connection with the GHP Formulation

Now that we have performed the same analysis for a generic type D background and Schwarzschild, we are in a position to make a direct comparison. For that purpose, we introduce a complete set of null tetrad vectors:

$$l^a = (1/f(r), 1, 0, 0), \quad n^a = \frac{1}{2}(1, -f(r), 0, 0), \quad m^a = \frac{1}{\sqrt{2}r}(0, 0, 1, i/\sin\theta). \tag{39}$$

With this choice (introduced by Kinnersley[11]), the tetrad components of the residual gauge vector in Schwarzschild become, after inserting (34) and (35) into (30) and (31):

$$\begin{aligned}
\xi_l &= \xi_a l^a = e^{-i\omega(t-r_*)} C_2 Y_{\ell m}(\theta,\phi), \\
\xi_n &= \xi_a n^a = -e^{-i\omega(t-r_*)}[C_1 - C_2(ir\omega + f(r)/2)] Y_{\ell m}(\theta,\phi), \\
\xi_m &= \xi_a m^a = \frac{1}{\sqrt{2}} e^{-i\omega(t-r_*)}[C_2 + rC_3 + irD]\left[\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right] Y_{\ell m}(\theta,\phi).
\end{aligned} \tag{40}$$

Before carrying out the comparison, we need to express the null derivative operators in the Kinnersley tetrad (39). Acting on a scalar quantity, the operator Þ is given by

$$Þ = l^a \Theta_a = \frac{1}{f(r)}\frac{\partial}{\partial t} + \frac{\partial}{\partial r}. \tag{41}$$

Therefore, in Schwarzschild, quantities annihilated by Þ have the the form

$$x^\circ = x^\circ(t - r_*, \theta, \phi). \tag{42}$$

Also, in Schwarzschild $\bar\rho = \rho = -1/r$ and $\tau = \tau' = 0$. So, when acting on scalars of spin $s\ (= (p-q)/2)$, the operators $\tilde{ð}$ and $\tilde{ð}'$ are (see (B.6) and (B.7) in Appendix B, and [5])

$$\begin{aligned}
\tilde{ð} &= \frac{ð}{\bar\rho} = -r m^a \Theta_a = -\frac{(\sin\theta)^s}{\sqrt{2}}\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big)(\sin\theta)^{-s}, \\
\tilde{ð}' &= \frac{ð'}{\rho} = -r\bar m^a \Theta_a = -\frac{(\sin\theta)^{-s}}{\sqrt{2}}\Big(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big)(\sin\theta)^{s}.
\end{aligned} \tag{43}$$

Also, since $\bar\Psi_2 = \Psi_2 = \Psi^\circ\rho^3 = -M/r^3$, we can write (see (B.5) in Appendix B, and [4])

$$\begin{aligned}
\tilde{Þ}' &= Þ' + \frac{(p+q)\Psi_2}{2\rho} = n^a\Theta_a + \frac{1}{2}(p+q)\Psi^\circ\rho^2 \\
&= \frac{f(r)^{-(p+q)/2}}{2}\Big(\frac{\partial}{\partial t} - f(r)\frac{\partial}{\partial r} + (p+q)\frac{M}{r^2}\Big) f(r)^{(p+q)/2}.
\end{aligned} \tag{44}$$

Now, given that, in Schwarzschild, $\tau^\circ = \pi^\circ = \alpha^\circ = 0$ and $\Psi^\circ = M$ we can write (17) as

$$\begin{aligned}
\xi_l &= \xi_l^\circ, \\
\xi_n &= \xi_n^\circ - \frac{r}{2}\Big(\frac{\partial}{\partial t} - f(r)\frac{\partial}{\partial r} + \frac{2M}{r^2}\Big)\xi_l^\circ, \\
\xi_m &= -r\xi_m^\circ + \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big)\xi_l^\circ,
\end{aligned} \tag{45}$$

where we have used $p = q = 1$ in (44). Comparing (40) and (45), it is clear that††

$$\begin{aligned}
\xi_l^\circ &= e^{-i\omega(t-r_*)} C_2 Y_{\ell m}(\theta,\phi), \\
\xi_n^\circ &= -e^{-i\omega(t-r_*)}(C_1 - C_2/2) Y_{\ell m}(\theta,\phi), \\
\xi_m^\circ &= -e^{-i\omega(t-r_*)}\frac{C_3 + iD}{\sqrt{2}}\Big(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\Big) Y_{\ell m}(\theta,\phi).
\end{aligned} \tag{46}$$

To conclude, by comparing (38) and (16), and recalling $\bar\rho = \rho = -1/r$, we see that with

$$A = a^\circ + \bar a^\circ, \quad \text{and} \quad B = -2b^\circ, \tag{47}$$

the equivalence of the two formulations is established. Finally, with $\rho'^\circ = -\frac{1}{2}$, and using (B.8), (46) and (27) in (13), we can demonstrate its complete correspondence with (36).

†† To obtain these relations, we have used $rf'(r) = 1 - f(r)$, which holds in the Schwarzschild spacetime.

7. Spherical symmetry and $\ell = 0$

Throughout sections 5 and 6 we have frequently used the word "generally" to describe our ability to find solutions for, and cancel, the trace of a vacuum perturbation. We now explain why the respective statements were not a little stronger. First note that in (36), when $l = 0$, two of the four terms vanish identically. Next note that for $\omega = 0$, another term also vanishes. It would appear that, for a static mass perturbation, we can no longer cancel the A-term from the trace in (38). This is a mere artifact of our analysis and is not a fundamental obstacle. Use of the Fourier transform in t is not permitted unless the perturbations belong in some suitable function space, say L^2. Polynomials would fail this test, but they are required in the gauge transformation for this case[18]. Allowing polynomials in t and restoring the time derivative in (36) corrects this defect.

Our use of "generally" in section 5 is somewhat more technical. For most type D spacetimes, ρ is complex and the analysis proceeds as indicated, except for the lowest $s = 0$ mode of a perturbation. It appears that, in this case, there is not enough gauge freedom to remove the trace from $\mathcal{E}_{ll} = 0$ in a general type D spacetime. However, for spherical symmetry (including Taub-Nut), ρ is real and a number of changes take place. First, from (14), we see that the solution for $h_{m\bar{m}}$ is now changed. It turns out that the remaining gauge freedom in (18) is also changed, since the terms involving $\tilde{\eth}$ and $\tilde{\eth}'$ vanish for $s = 0$. However, with the choice $\xi_l{}^{\circ} \neq 0$, the residual gauge freedom is again sufficient to permit complete removal of the trace. This is consistent with the explicit discussion given in the previous paragraph for the Schwarzschild spacetime.

8. Discussion

We have concentrated on Petrov type II spacetimes in this paper because they satisfy a minimum requirement necessary for the existence of a radiation gauge, namely the occurrence of a repeated PND. As is common in the use of NP methods, we have been able to do this without either choosing coordinates or finding a metric. The Held technique has allowed us to partially integrate the perturbation equations in just such a context, for example, to solve the equation $\mathcal{E}_{ll} = 0$ while investigating circumstances for the existence of an IRG. Furthermore, in type D, it has allowed us to completely characterize the residual gauge freedom, and use it in the radiation gauge construction.

For perturbations with $\mathcal{T}_{ll} = 0$, we have found that type II spacetimes allow one to "impose" a radiation gauge, in the sense that one can set $h_{m\bar{m}} = 0$ as a (trivial) solution of $\mathcal{E}_{ll} = 0$. It is arguable whether this should be called a gauge choice at all, rather than a choice of solution. Without further work which is beyond our present scope, that is the best we can do for this class of spacetime. For type D spacetimes, our characterization of the residual gauge freedom is sufficiently complete that we can explicitly demonstrate the required gauge choice to generally remove any non-zero solution for the trace obtained via $\mathcal{E}_{ll} = 0$. Thus, in the type D case, radiation gauges can be established by a genuine gauge choice, even if only after a solution of $\mathcal{E}_{ll} = 0$ is chosen.

It is interesting to compare the subtle differences between the type II and type D cases with other differences which can be identified in the construction of Hertz potentials for the two cases. Stewart[22] writes out the type II case rather fully for an IRG. Thus, in this case, the perturbation in Ψ_0 is tetrad and gauge invariant, while the potential satisfies the adjoint (in the sense detailed by Wald[24]) of the $s = +2$ Teukolsky equation. Remarkably, in the type D case, this adjoint is actually the $s = -2$ Teuklosky equation, also satisfied by the gauge and tetrad invariant perturbation in Ψ_4. In the type II case, the adjoint equation is the same as in type D, but Ψ_4 is no longer tetrad invariant. Compared to the type D result, the expression for Ψ_4 given by Stewart has many extra terms depending on κ' and σ', so presumably it does not satisfy the same equation as the potential and, as a consequence, metric reconstruction would be restricted to being built around the perturbation for Ψ_0.

We cannot yet tell if it is this difference which is reflected in the difference we observe (between the type II and type D cases) for the use of gauge in the construction of an IRG. Either way, there are very interesting parallels here. Stewart does not really address the question of gauges, but does address problem of potentials. We do it the other way around, and show that the results for existence bear very close correspondence. In fact, they are essentially the same. The current need to choose $h_{m\bar{m}} = 0$ in type II, rather than being able to set it so by (residual) gauge freedom, appears to be the only difference we observe. For Stewart, the type II potential exists, but it appears unrelated to any equation for Ψ_4. This is not so in type D where radiation gauges really exist.

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Appendix A. The Perturbed Einstein Equations for a type II Background

In this appendix we write the EEs for an arbitrary type II background. We've assumed the PND is aligned with l^a and made use of the Goldberg-Sachs theorem. Note that the equations for $\mathcal{E}_{lm}$, $\mathcal{E}_{nm}$ and $\mathcal{E}_{mm}$ are complex, so $\mathcal{E}_{l\bar{m}} = \bar{\mathcal{E}}_{lm}$ and so on:

$$\begin{aligned}\mathcal{E}_{ll} = {} & \{(\eth' - \tau')(\eth - \bar{\tau}') + \rho(\text{Þ}' + \rho' - \bar{\rho}') - (\text{Þ} - \rho)\rho' + \Psi_2\}h_{ll} \\ & + \{-(\rho + \bar{\rho})(\text{Þ} + \rho + \bar{\rho}) + 4\rho\bar{\rho}\}hln \\ & + \{-(\text{Þ} - 3\bar{\rho})(\eth' - \tau' + \bar{\tau}) + \bar{\tau}\text{Þ} - \bar{\rho}\eth'\}h_{lm} \\ & + \{-(\text{Þ} - 3\rho)(\eth + \tau - \bar{\tau}') + \tau\text{Þ} - \rho\eth\}h_{l\bar{m}} \\ & + \{\text{Þ}(\text{Þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}},\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
\mathcal{E}_{nn} = {} & \{2\kappa'\bar{\kappa}'\}h_{ll} \\
& + \{(\eth' - \bar{\tau})(\eth - \tau) + \bar{\rho}'(\text{Þ} - \rho + \bar{\rho}) - (\text{Þ}' - \bar{\rho}')\bar{\rho} + \bar{\Psi}_2 + 2\bar{\rho}\bar{\rho}'\}h_{nn} \\
& + \{-(\rho' + \bar{\rho}')(\text{Þ}' + \rho' + \bar{\rho}') + 4\rho'\bar{\rho}' - (\eth' - 2\bar{\tau})\bar{\kappa}' - (\eth - 2\tau)\kappa'\}h_{ln} \\
& + \{(\text{Þ}' - \bar{\rho}')\kappa' + \kappa'(\text{Þ}' - \rho' - \bar{\rho}') - \bar{\kappa}'\sigma'\}h_{lm} \\
& + \{(\text{Þ}' - \rho')\bar{\kappa}' + \bar{\kappa}'(\text{Þ}' - \bar{\rho}' - \rho') - \kappa'\bar{\sigma}'\}h_{l\bar{m}} \\
& + \{-(\text{Þ}' - 3\rho')(\eth' + \tau' - \bar{\tau}) + \tau'\text{Þ}' - \rho'\eth' - \kappa'\text{Þ} \\
& \quad + (\text{Þ} - 2\rho + \bar{\rho})\kappa' + (\eth - 3\tau + \bar{\tau}')\sigma' + \eth(\sigma') - \Psi_3\}h_{nm} \\
& + \{-(\text{Þ}' - 3\bar{\rho}')(\eth + \bar{\tau}' - \tau) + \bar{\tau}'\text{Þ}' - \bar{\rho}'\eth - \bar{\kappa}'\text{Þ} \\
& \quad + (\text{Þ} - 2\bar{\rho} + \rho)\bar{\kappa}' + (\eth' - 3\bar{\tau} + \tau')\bar{\sigma}' + \eth'(\bar{\sigma}') - \bar{\Psi}_3\}h_{n\bar{m}} \\
& + \{-(\eth' - 2\bar{\tau})\kappa' - \sigma'(\text{Þ}' - \rho' + \bar{\rho}')\}h_{mm} \\
& + \{-(\eth - 2\tau)\bar{\kappa}' - \bar{\sigma}'(\text{Þ}' - \bar{\rho}' + \rho')\}h_{\bar{m}\bar{m}} \\
& + \{\text{Þ}'(\text{Þ}' - \rho' - \bar{\rho}') + \kappa'(\tau - \bar{\tau}') + \bar{\kappa}'(\bar{\tau} - \tau') + 2\sigma'\bar{\sigma}' + 2\rho'\bar{\rho}'\}h_{m\bar{m}},
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
\mathcal{E}_{ln} = {} & \tfrac{1}{2}\{\rho'(\text{Þ}' - \rho') + \bar{\rho}'(\text{Þ}' - \bar{\rho}') + (\eth - 2\bar{\tau}')\kappa' + (\eth' - 2\tau')\bar{\kappa}' + 2\sigma'\bar{\sigma}'\}h_{ll} \\
& + \tfrac{1}{2}\{\rho(\text{Þ} - \rho) + \bar{\rho}(\text{Þ} - \bar{\rho})\}h_{nn} \\
& + \tfrac{1}{2}\{-(\eth' + \tau' + \bar{\tau})(\eth - \tau - \bar{\tau}') - (\eth'\eth + 3\tau\tau' + 3\bar{\tau}\bar{\tau}') + 2(\bar{\tau} + \tau')\eth \\
& \quad + (\text{Þ} - 2\bar{\rho})\rho' + (\text{Þ}' - 2\rho')\bar{\rho} - \bar{\rho}'(\text{Þ} + \rho) - \rho(\text{Þ}' + \bar{\rho}') - \Psi_2 - \bar{\Psi}_2\}h_{ln} \\
& + \tfrac{1}{2}\{(\text{Þ}' - 2\bar{\rho}')(\eth' - \tau') + \bar{\tau}(\text{Þ}' + \rho' + \bar{\rho}') - \tau'(\text{Þ}' - \rho') \\
& \quad - (2\eth' - \bar{\tau})\bar{\rho}' - (\text{Þ} - 2\bar{\rho})\kappa' + \sigma'(\tau - \bar{\tau}')\}h_{lm} \\
& + \tfrac{1}{2}\{(\text{Þ}' - 2\rho')(\eth - \bar{\tau}') + \tau(\text{Þ}' + \bar{\rho}' + \rho') - \bar{\tau}'(\text{Þ}' - \bar{\rho}') \\
& \quad - (2\eth - \tau)\rho' - (\text{Þ} - 2\rho)\bar{\kappa}' + \bar{\sigma}'(\bar{\tau} - \tau')\}h_{l\bar{m}} \\
& + \tfrac{1}{2}\{(\text{Þ} - 2\rho)(\eth' - \bar{\tau}) + (\tau' + \bar{\tau})(\text{Þ} + \bar{\rho}) - 2(\eth' - \tau')\rho - 2\bar{\tau}\text{Þ}\}h_{nm} \\
& + \tfrac{1}{2}\{(\text{Þ} - 2\bar{\rho})(\eth - \tau) + (\bar{\tau}' + \tau)(\text{Þ} + \rho) - 2(\eth - \bar{\tau}')\bar{\rho} - 2\tau\text{Þ}\}h_{n\bar{m}} \\
& + \tfrac{1}{2}\{-(\eth' - \bar{\tau})(\eth' - \tau') + \bar{\tau}(\bar{\tau} - \tau') - \sigma'\rho\}h_{mm} \\
& + \tfrac{1}{2}\{-(\eth - \tau)(\eth - \bar{\tau}') + \tau(\tau - \bar{\tau}') - \bar{\sigma}'\bar{\rho}\}h_{\bar{m}\bar{m}} \\
& + \tfrac{1}{2}\{(\eth' + \tau' - \bar{\tau})(\eth - \tau + \bar{\tau}') + (\eth'\eth - \tau\tau' - \bar{\tau}\bar{\tau}' + \tau\bar{\tau}) - (\Psi_2 + \bar{\Psi}_2) \\
& \quad + (\text{Þ}' - 2\rho')\bar{\rho} + (\text{Þ} - 2\bar{\rho})\rho' + \rho(3\text{Þ}' - 2\bar{\rho}') + \bar{\rho}'(3\text{Þ} - 2\rho) \\
& \quad - 2\text{Þ}'\text{Þ} + 2\rho\bar{\rho}' + 2\eth'(\tau) - \tau\bar{\tau}\}h_{m\bar{m}},
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\mathcal{E}_{lm} = {}& \tfrac{1}{2}\{(\text{Þ}' - \rho')(\eth - \bar{\tau}') + (\eth - \tau - 2\bar{\tau}')\bar{\rho}' - (\eth - \tau)\rho' + \tau(\text{Þ}' + \rho') \\
& \quad + \bar{\sigma}'(\eth' - \tau' + \bar{\tau}) + \bar{\Psi}_3 + \bar{\rho}\bar{\kappa}'\}h_{ll} \\
& + \tfrac{1}{2}\{-(\text{Þ} - \rho + \bar{\rho})(\eth + \tau - \bar{\tau}') - (\eth - 3\tau + \bar{\tau}')\bar{\rho} - 2\rho\bar{\tau}'\}h_{ln} \\
& + \tfrac{1}{2}\{-(\text{Þ}' + \bar{\rho}')(\text{Þ} - 2\bar{\rho}) + \rho(\text{Þ}' + 2\rho' - 2\bar{\rho}') - 4\rho'\bar{\rho} + 2\Psi_2 \\
& \quad + (\eth' + \bar{\tau})(\eth - 2\bar{\tau}') - \tau(\eth' + \tau' - 2\bar{\tau}) - \tau'(\tau - 4\bar{\tau}')\}h_{lm} \\
& + \tfrac{1}{2}\{-\eth(\eth - 2\tau) - \bar{\sigma}'(\text{Þ} + 2\bar{\rho} - 4\rho) - 2\bar{\tau}'(\tau - \bar{\tau}')\}h_{l\bar{m}} \\
& + \tfrac{1}{2}\{\text{Þ}(\text{Þ} - 2\rho) + 2\bar{\rho}(\rho - \bar{\rho})\}h_{nm} \\
& + \tfrac{1}{2}\{-(\text{Þ} - \bar{\rho})(\eth' - \tau' + \bar{\tau}) + 2\bar{\tau}\bar{\rho}\}h_{mm} \\
& + \tfrac{1}{2}\{(\text{Þ} + \rho - \bar{\rho})(\eth + \bar{\tau}' - \tau) + 2\bar{\tau}'(\text{Þ} - 2\rho) - (\eth - \tau - \bar{\tau}')\bar{\rho} + 2\rho\tau\}h_{m\bar{m}},
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
\mathcal{E}_{n\bar{m}} = {}& \tfrac{1}{2}\{(\text{Þ}' - \rho')\kappa' + \kappa'\text{Þ}' + \bar{\kappa}'\sigma'\}h_{ll} \\
& + \tfrac{1}{2}\{(\text{Þ} - \rho + \bar{\rho})(\eth' - \bar{\tau}) - (\eth' - 2\tau' + \bar{\tau})\rho + \tau'(\text{Þ} - \bar{\rho})\}h_{nn} \\
& + \tfrac{1}{2}\{(-(\text{Þ}' - \rho' + \bar{\rho}')(\eth' + \tau' - \bar{\tau}) - (\eth' - 3\tau' + \bar{\tau})\bar{\rho}' + (\eth - \tau + \bar{\tau}')\sigma' \\
& \quad - 2\sigma'\eth - \Psi_3 - 2\rho'\bar{\tau}\}h_{ln} \\
& + \{\sigma'(\rho' - 2\bar{\rho}') - \kappa'(\tau' - 2\bar{\tau}) + \tfrac{1}{2}\Psi_4\}h_{lm} \\
& + \tfrac{1}{2}\{(\text{Þ}'(\text{Þ}' - 2\rho') + -\kappa'(\eth - 2\tau + 2\bar{\tau}') + \bar{\kappa}'(\eth' - 4\tau' + 2\bar{\tau}) \\
& \quad + 2\bar{\rho}'(\rho' - \bar{\rho}') + 2\sigma'\bar{\sigma}'\}h_{l\bar{m}} \\
& + \tfrac{1}{2}\{-\eth'(\eth' - 2\tau') + \sigma'(\text{Þ} - 2\rho + 2\bar{\rho}) - 2\bar{\tau}(\tau' - \bar{\tau})\}h_{nm} \\
& + \tfrac{1}{2}\{-(\text{Þ}' - \bar{\rho}')(\text{Þ} + 2\bar{\rho}) + \rho(\text{Þ}' - 2\bar{\rho}') + 2\bar{\rho}'(\text{Þ} - \rho) - \Psi_2 - 2\bar{\Psi}_2 \\
& \quad + (\eth' - 3\bar{\tau})\eth + \bar{\tau}'(2\eth' - \bar{\tau} + 4\tau') - \tau(\eth' - 2\bar{\tau})\}h_{n\bar{m}} \\
& + \tfrac{1}{2}\{-(\eth' - \tau')\sigma' - \sigma'\eth'\}h_{mm} \\
& + \tfrac{1}{2}\{-(\text{Þ}' - \bar{\rho}')(\eth - \tau + \bar{\tau}') + 2\bar{\tau}'\bar{\rho}' - \bar{\kappa}'(\text{Þ} - 2\rho + 2\bar{\rho}) + \eth'(\bar{\sigma}') - \bar{\tau}\bar{\sigma}'\}h_{\bar{m}\bar{m}} \\
& + \tfrac{1}{2}\{(\text{Þ}' + \rho' - \bar{\rho}')(\eth' - \tau' + \bar{\tau}) + 2\bar{\tau}(\text{Þ}' - 2\rho') - (\eth' - \tau' - \bar{\tau})\bar{\rho}' + 2\rho'\tau' \\
& \quad + (\eth - \tau - \bar{\tau}')\sigma' + \sigma'\eth - \kappa'\text{Þ} - \Psi_3\}h_{m\bar{m}},
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\mathcal{E}_{mm} = {}& \{(\text{Þ}' - 2\rho')\bar{\sigma}' + \bar{\kappa}'(\eth + \tau - \bar{\tau}')\}h_{ll} \\
& + \{-\eth(\eth - \tau - \bar{\tau}') - 2\tau\bar{\tau}' + \bar{\sigma}'(\rho - \bar{\rho})\}h_{ln} \\
& + \{(\text{Þ}' - \rho')(\eth - \bar{\tau}') - (\eth - \tau - \bar{\tau}')\rho' + \tau(\text{Þ}' + \rho' - \bar{\rho}') - (\text{Þ} - 2\bar{\rho})\bar{\kappa}' \\
& \quad - \bar{\tau}'(\text{Þ} + \bar{\rho}') + \bar{\tau}\bar{\sigma}' - \bar{\Psi}_3\}h_{lm} \\
& + \{-(\eth - \tau - \bar{\tau}')\bar{\sigma}' - \bar{\sigma}'(\eth - \tau)\}h_{l\bar{m}} \\
& + \{(\text{Þ} - \bar{\rho})(\eth - \tau) - (\eth - \tau - \bar{\tau}')\bar{\rho} - \tau(\text{Þ} + \rho) + \bar{\tau}'(\text{Þ} - \rho + \bar{\rho})\}h_{nm} \\
& + \{-(\text{Þ}' - \rho')(\text{Þ} - \bar{\rho}) + (\eth - \tau)\tau' - \tau(\eth' + \tau' - \bar{\tau}) + \Psi_2\}h_{mm} \\
& + \{(\text{Þ} - 2\bar{\rho})\bar{\sigma}' + (\tau + \bar{\tau}')\eth + (\tau - \bar{\tau}')^2\}h_{m\bar{m}},
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\mathcal{E}_{m\bar{m}} = {} & \tfrac{1}{2}\{\text{Þ}'(\text{Þ}' - \rho' - \bar{\rho}') + 2\rho'\bar{\rho}' + \kappa'(\tau - \bar{\tau}') - \bar{\kappa}'(\bar{\tau} - \tau') + 2\sigma'\bar{\sigma}'\}h_{ll} \\
& + \tfrac{1}{2}\{\text{Þ}(\text{Þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{nn} \\
& + \tfrac{1}{2}\{-(\text{Þ}' + \rho' - \bar{\rho}')(\text{Þ} - \rho + \bar{\rho}) - \text{Þ}'(\text{Þ} + \rho) + \rho(\text{Þ}' + \rho' - \bar{\rho}') - \bar{\Psi}_2 \\
& \quad + (\eth' - \bar{\tau})(\eth - \tau - \bar{\tau}') + \eth'\eth - (\eth - 2\bar{\tau}')\tau' - \bar{\tau}(2\eth + \bar{\tau}') \\
& \quad - 2\tau(\eth' - \bar{\tau}) + 2\tau'\bar{\tau}' + \bar{\rho}\bar{\rho}'\}h_{ln} \\
& + \tfrac{1}{2}\{-(\text{Þ}' - 2\rho')(\eth' - 2\bar{\tau}) + \bar{\tau}(\text{Þ}' + 2\rho' - 2\bar{\rho}') + 2(\eth - \bar{\tau}')\sigma' - \sigma'\eth \\
& \quad - 2\tau'\bar{\rho}' - 2\kappa'(\rho - \bar{\rho}) - \Psi_3\}h_{lm} \\
& + \tfrac{1}{2}\{-(\text{Þ}' - 2\bar{\rho}')(\eth - 2\tau) + \tau(\text{Þ}' + 2\bar{\rho}' - 2\bar{\rho}) + 2(\eth' - \tau')\bar{\sigma}' - \bar{\sigma}'\eth' \\
& \quad - 2\bar{\tau}'\rho' - 2\bar{\kappa}'(\bar{\rho} - \rho) - \bar{\Psi}_3\}h_{l\bar{m}} \\
& + \tfrac{1}{2}\{-(\text{Þ} - 2\bar{\rho})(\eth' - 2\tau') + \tau'(\text{Þ} - 2\rho - 2\bar{\rho}) - 2\rho\bar{\tau} + 4\tau'\bar{\rho}\}h_{nm} \\
& + \tfrac{1}{2}\{-(\text{Þ} - 2\rho)(\eth - 2\bar{\tau}') + \bar{\tau}'(\text{Þ} - 2\bar{\rho} - 2\rho) - 2\bar{\rho}\tau + 4\bar{\tau}'\rho\}h_{n\bar{m}} \\
& + \tfrac{1}{2}\{-\bar{\tau}(\eth' - \bar{\tau}) - \tau'(\eth' - \tau') - (\text{Þ} - 2\bar{\rho})\sigma'\}h_{mm} \\
& + \tfrac{1}{2}\{-\tau(\eth - \tau) - \bar{\tau}'(\eth - \bar{\tau}') - (\text{Þ} - 2\rho)\bar{\sigma}'\}h_{\bar{m}\bar{m}}.
\end{aligned}
\tag{A.7}
$$

Appendix B. Integration à la Held[7, 21]

We provide details of the integration that lead to (16) and (17). We will need some results (and their complex conjugates) from the integration of the type D background:

$$\tilde{\eth}'\rho = -\pi^\circ \frac{\rho}{\bar{\rho}} - \alpha^\circ\rho - \bar{\tau}^\circ\rho^2, \tag{B.1}$$

$$\tau = -\bar{\pi}^\circ - \bar{\alpha}^\circ\rho + \tau^\circ\rho\bar{\rho}, \tag{B.2}$$

$$\tau' = -\pi^\circ - \bar{\tau}^\circ\rho^2, \tag{B.3}$$

$$\Psi_2 = \Psi^\circ\rho^3. \tag{B.4}$$

As noted in the text, $\pi^\circ \neq 0$ leads to the accelerating C-metrics, which we include for full generality. We will also need the definitions of the new operators:

$$\tilde{\text{Þ}}' = \text{Þ}' - \bar{\tau}\tilde{\eth} - \tau\tilde{\eth}' + \tau\bar{\tau}(\frac{p}{\bar{\rho}} + \frac{q}{\rho}) + \tfrac{1}{2}(\frac{p\Psi_2}{\rho} + \frac{q\bar{\Psi}_2}{\bar{\rho}}), \tag{B.5}$$

$$\tilde{\eth} = \frac{\eth}{\bar{\rho}} + \frac{q\tau}{\rho}, \tag{B.6}$$

$$\tilde{\eth}' = \frac{\eth'}{\rho} + \frac{p\bar{\tau}}{\bar{\rho}}, \tag{B.7}$$

where p and q label the GHP type of the quantity being acted on (see [4] or [25]). To evaluate (13), we will also need one more result from integration on the background:

$$\rho' = \rho'^\circ\bar{\rho} - \tfrac{1}{2}\Psi^\circ\rho^2 - (\tilde{\eth}\bar{\tau}^\circ + \tfrac{1}{2}\Psi^\circ)\rho\bar{\rho} - \tau^\circ\bar{\tau}^\circ\rho^2\bar{\rho}. \tag{B.8}$$

We are now ready to begin the integration. We'll start with $\mathcal{E}_{ll} = 0$, which we can rewrite, with the help of $\text{Þ}\rho = \rho^2$ and its complex conjugate, as

$$\rho^2\text{Þ}\Big[\frac{\bar{\rho}}{\rho^3}\text{Þ}\Big(\frac{\rho}{\bar{\rho}}h_{m\bar{m}}\Big)\Big] = 0. \tag{B.9}$$

Integrating once gives

$$Þ\Big(\frac{\rho}{\bar\rho}h_{m\bar m}\Big) = b^\circ\frac{\rho^3}{\bar\rho}, \tag{B.10}$$

and another integration leads to

$$h_{m\bar m} = \bar a^\circ\frac{\bar\rho}{\rho} + \tfrac12 b^\circ(\rho+\bar\rho). \tag{B.11}$$

However, $h_{m\bar m}$ is, by definition, a real quantity, so we add the complex conjugate and use b° to represent a real quantity in the second term. The final result is that

$$h_{m\bar m} = a^\circ\frac{\rho}{\bar\rho} + \bar a^\circ\frac{\bar\rho}{\rho} + b^\circ(\rho+\bar\rho). \tag{B.12}$$

We turn next to the equations governing the residual gauge freedom, beginning with

$$Þ\xi_l = 0, \tag{B.13}$$

which integrates trivially to give

$$\xi_l = \xi_l{}^\circ. \tag{B.14}$$

With this information in hand, we can now integrate the equation governing ξ_m:

$$(Þ+\bar\rho)\xi_m + (ð+\bar\tau')\xi_l = 0. \tag{B.15}$$

Rewriting the $Þ$ piece and using (B.6) with $p=1$ leads to

$$\frac{1}{\bar\rho}Þ(\bar\rho\xi_m) + \bar\tau'\xi_l + \rho\tilde{ð}\xi_l - \frac{\bar\rho\tau}{\rho}\xi_l = 0, \tag{B.16}$$

which, after substituting (B.2), the complex conjugate of (B.3) and (B.14) along with some rearranging, yields

$$Þ(\bar\rho\xi_m) = -\bar\pi^\circ\xi_l{}^\circ\Big(\frac{\bar\rho^2}{\rho}-\bar\rho\Big) + 2\tau^\circ\xi_l{}^\circ\bar\rho^3 - \bar\rho^2(\tilde{ð}+\bar\alpha^\circ)\xi_l{}^\circ. \tag{B.17}$$

Integration then gives us

$$\xi_m = \xi_m{}^\circ\frac{1}{\bar\rho} - \bar\pi^\circ\xi_l{}^\circ\frac{1}{\rho} + \tau^\circ\xi_l{}^\circ\bar\rho - (\tilde{ð}+\bar\alpha^\circ)\xi_l{}^\circ, \tag{B.18}$$

and the solution for $\xi_{\bar m}$ then follows from complex conjugation

$$\xi_{\bar m} = \xi_{\bar m}{}^\circ\frac{1}{\rho} - \pi^\circ\xi_l{}^\circ\frac{1}{\bar\rho} + \bar\tau^\circ\xi_l{}^\circ\rho - (\tilde{ð}'+\alpha^\circ)\xi_l{}^\circ. \tag{B.19}$$

Finally, we are in a position to deal with ξ_n, by writing

$$Þ'\xi_l + Þ\xi_n + (\tau+\bar\tau')\xi_{\bar m} + (\bar\tau+\tau')\xi_m = 0, \tag{B.20}$$

in terms of Held's operators ((B.1), (B.2) and (B.3)) as

$$\begin{aligned} &Þ\xi_n + \tilde{Þ}'\xi_l + \bar\tau\tilde{ð}\xi_l + \tau\tilde{ð}'\xi_l - \tau\bar\tau\Big(\frac{1}{\rho}+\frac{1}{\bar\rho}\Big)\xi_l \\ &\quad - \tfrac12\Big(\frac{\Psi_2}{\rho}+\frac{\bar\Psi_2}{\bar\rho}\Big)\xi_l + (\tau+\bar\tau')\xi_{\bar m} + (\bar\tau+\tau')\xi_m = 0. \end{aligned} \tag{B.21}$$

Substituting (B.2), (B.3), (B.4), (B.14), (B.18) and (B.19), rearranging terms and letting the dust settle leads to

$$
\begin{aligned}
\text{Þ}\xi_n = &-\tilde{\text{Þ}}'\xi_l{}^\circ + \tfrac{1}{2}\Psi^\circ\xi_l{}^\circ\rho^2 + \tfrac{1}{2}\bar{\Psi}^\circ\xi_l{}^\circ\bar{\rho}^2 - \pi^\circ\bar{\pi}^\circ\xi_l{}^\circ\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right) \\
&+ \tau^\circ\bar{\tau}^\circ\xi_l{}^\circ(\rho^2\bar{\rho} + \rho\bar{\rho}^2) - [\bar{\tau}^\circ\rho^2(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ\bar{\rho}^2(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ \\
&- [\pi^\circ(\tilde{\eth} + \bar{\alpha}^\circ) + \bar{\pi}^\circ(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ + 2\pi^\circ\xi_m{}^\circ\frac{1}{\bar{\rho}} + 2\bar{\pi}^\circ\xi_{\bar{m}}{}^\circ\frac{1}{\rho} \\
&+ \tau^\circ\xi_{\bar{m}}{}^\circ\left(\frac{\bar{\rho}^2}{\rho} - \bar{\rho}\right) + \bar{\tau}^\circ\xi_m{}^\circ\left(\frac{\rho^2}{\bar{\rho}} - \rho\right) + \alpha^\circ\xi_m{}^\circ + \bar{\alpha}^\circ\xi_{\bar{m}}{}^\circ.
\end{aligned}
\tag{B.22}
$$

Integration then results in

$$
\begin{aligned}
\xi_n = &\,\xi_n{}^\circ + \tfrac{1}{2}\Psi^\circ\xi_l{}^\circ\rho + \tfrac{1}{2}\bar{\Psi}^\circ\xi_l{}^\circ\bar{\rho} + \tau^\circ\bar{\tau}^\circ\xi_l{}^\circ\rho\bar{\rho} + \tfrac{1}{2}\pi^\circ\bar{\pi}^\circ\xi_l{}^\circ\left(\frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2}\right) \\
&+ \left[\frac{\pi^\circ}{\rho}(\tilde{\eth} + \bar{\alpha}^\circ) + \frac{\bar{\pi}^\circ}{\bar{\rho}}(\tilde{\eth}' + \alpha^\circ)\right]\xi_l{}^\circ + \tfrac{1}{2}\left(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\right)\tilde{\text{Þ}}'\xi_l{}^\circ \\
&- [\bar{\tau}^\circ\rho(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ\bar{\rho}(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ + \bar{\tau}^\circ\xi_m{}^\circ\frac{\rho}{\bar{\rho}} + \tau^\circ\xi_{\bar{m}}{}^\circ\frac{\bar{\rho}}{\rho} \\
&- \pi^\circ\xi_m{}^\circ\frac{1}{\bar{\rho}^2} - \bar{\pi}^\circ\xi_{\bar{m}}{}^\circ\frac{1}{\rho^2} - \alpha^\circ\xi_m{}^\circ\frac{1}{\bar{\rho}} - \bar{\alpha}^\circ\xi_{\bar{m}}{}^\circ\frac{1}{\rho},
\end{aligned}
\tag{B.23}
$$

and our task is complete.

Appendix C. The Pertubed Einstein Tensor in Schwarzschild

We list the components of the Einstein tensor expressed in terms of the metric perturbations in the gauge $l^a h_{ab} = 0$. The independent components of the even parity metric perturbations in this gauge are, using (23): $H(r) \equiv H_0(r)$, $h(r) \equiv h_0(r)$, $K(r)$ and $G(r)$. The trace is given by $T(r) = 2K(r) - \ell(\ell+1)G(r)$. The components of the Einstein tensor $E_1 ... E_7$ are calculated to be:

$$
\begin{aligned}
E_1(r) = &\frac{(r-2M)}{2r}T''(r) + \frac{(3r-5M)}{2r^2}T'(r) - \frac{[\ell(\ell+1)-2]}{2r^2}K(r) \\
&- \frac{\ell(\ell+1)}{r^2}h'(r) - \frac{\ell(\ell+1)(r-3M)}{(r-2M)r^3}h(r) \\
&- \frac{(r-2M)}{r^2}H'(r) - \frac{[\ell(\ell+1)+2]}{2r^2}H(r),
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
E_2(r) = &-\frac{i\omega}{2}T'(r) - \frac{i\omega(r-3M)}{2r(r-2M)}T(r) + \frac{[\ell(\ell+1)+2ir\omega]}{2r^2}H(r) \\
&+ \frac{\ell(\ell+1)}{2r^2}h'(r) + \frac{\ell(\ell+1)(i\omega r^2-2M)}{2(r-2M)r^3}h(r),
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
E_3(r) = &-\frac{(r-M)}{2r^2}T'(r) - \frac{r\omega^2}{2(r-2M)}T(r) + \frac{\ell(\ell+1)(r-M-i\omega r^2)}{r^3(r-2M)}h(r) \\
&+ \frac{(r-2M)}{r^2}H'(r) - \frac{2ir\omega}{r^2}H(r) - \frac{[\ell(\ell+1)-2]}{2r^2}\left[H(r) - K(r)\right],
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
E_4(r) = {} & \frac{(r-2M)}{2r}h''(r) - \frac{i\omega}{2}h'(r) + \frac{[2M(r-2M) - i\omega r^2(r-3M)]}{r^3(r-2M)}h(r) \\
& + \frac{(r-2M)}{2r}H'(r) + \frac{(2M - i\omega r^2)}{2r^2}H(r) - \frac{i\omega}{2}\left[K(r) - G(r)\right],
\end{aligned}
\tag{C.4}
$$

$$
\begin{aligned}
E_5(r) = {} & -\frac{i\omega r}{2(r-2M)}h'(r) - \frac{[2(r-2M)(1-i\omega r) + r^3\omega^2]}{2r(r-2M)^2}h(r) \\
& - \frac{1}{2}H'(r) - \frac{(2M - i\omega r^2)}{2r(r-2M)}H(r) + \frac{1}{2}\left[K'(r) - G'(r)\right],
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
E_6(r) = {} & \frac{(r-2M)}{2r}H''(r) + \frac{(1-i\omega r)}{r}H'(r) - \frac{i\omega(2r - 2M - i\omega r^2)}{2r(r-2M)}H(r) \\
& + \ell(\ell+1)\Big[\frac{(r-2M)}{2r}G''(r) + \frac{(r-M)}{r^2}G'(r) + \frac{r\omega^2}{2(r-2M)}G(r)\Big] \\
& - \Big[\frac{(r-2M)}{2r}K''(r) + \frac{(r-M)}{r^2}K'(r) + \frac{r\omega^2}{2(r-2M)}K(r)\Big] \\
& + \frac{\ell(\ell+1)}{r^2}h'(r) - \frac{i\omega\ell(\ell+1)}{r(r-2M)}h(r),
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
E_7 = {} & \frac{(r-2M)}{2r}G''(r) + \frac{(r-M)}{r^2}G'(r) + \frac{r\omega^2}{2(r-2M)}G(r) \\
& + \frac{1}{r^2}h'(r) - \frac{i\omega}{r(r-2M)}h(r).
\end{aligned}
\tag{C.7}
$$

The independent components of the odd parity perturbations in this gauge are, using (22): $h(r) \equiv h_0(r)$ and $H(r) \equiv h_2(r)$. The components of Einstein tensor $F_1 ... F_3$ are calculated to be

$$
\begin{aligned}
F_1(r) = {} & \frac{r-2M}{2r}h''(r) - \frac{i\omega}{2}h'(r) + \frac{i\omega\left[\ell(\ell+1) - 2\right]}{4r^2}H(r) \\
& - \frac{i\omega r^3 + \left[\ell(\ell+1)/2 - i\omega 3M\right]r^2 - \left[\ell(\ell+1) + 2\right]rM + 4M^2}{r^3(r-2M)}h(r),
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
F_2(r) = {} & -\frac{\left[\ell(\ell+1) - 2\right]}{4r^2}H'(r) - \frac{i\omega r}{2(r-2M)}h'(r) + \frac{\left[\ell(\ell+1) - 2\right]}{2r^3}H(r) \\
& - \frac{r^3\omega^2/2 - (r-2M)\left[i\omega r - 1 + \ell(\ell+1)/2\right]}{r(r-2M)^2}h(r),
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
F_3(r) = {} & \frac{(r-2M)}{2r}H''(r) - \frac{(r-3M)}{r^2}H'(r) - h'(r) \\
& + \frac{i\omega r}{(r-2M)}h(r) + \frac{(r^4\omega^2/2 + r^2 - 6Mr + 8M^2)}{r^3(r-2M)}H(r).
\end{aligned}
\tag{C.10}
$$

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